Solutions For A Generalized Fractional Anomalous Diffusion Equation

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In this paper, we investigate the solutions for a generalized fractional diffusion equation that extends some known diffusion equations by taking a spatial time-dependent diffusion coefficient and an external force into account, which subjects to the natural boundaries and the generic initial condition. We obtain explicit analytical expressions for the probability distribution and study the relation between our solutions and those obtained within the maximum entropy principle by using the Tsallis entropy.

Keywords: Anomalous diffusion; Fractional diffusion; Green function; Fox function

I. INTRODUCTION

Recently, anomalous diffusion equations have been extensively investigated due to the broadness of their physical applications. In fact, fractional diffusion equations and the non-linear fractional diffusion equations have been successfully applied to several physical situations such as percolation of gases through porous media [1], thin saturated regions in porous media [2], in the transport of fluid in porous media and in viscous fingering[3], thin liquid films spreading under gravity [4], modeling of non-Markvian dynamical processes in protein folding [5], relaxation to equilibrium in system (such as polymer chains and membranes) with long temporal memory [6], anomalous transport in disordered systems [7], diffusion on fractals [8], and the multi-physical transport in porous media, such as electroosmosis[9-10]. Note that the physical systems mentioned above essentially concern anomalous diffusion of the correlated type (both sub and super-diffusion; see [11] and references therein) or of the Lévy type (see [12] and references therein). The anomalous correlated diffusion usually has a finite second moment $\langle x^2 \rangle \propto t^\sigma$ ($\sigma > 1, \sigma = 1$ and $0 < \sigma < 1$ correspond to super-diffusion, normal diffusion and sub-diffusion, respectively; $\sigma = 0$ corresponds basically to localization). Due to the broadness of the problems involving anomalous diffusion, one needs to apply different kinds of theoretical approaches such as nonlinear Fokker-Planck equation (or modified porous media equation), fractional Fokker-Planck equation, Fokker-Planck equation with spatial dependent diffusion coefficient , and generalized Langevin equations. The properties concerning these equations have been intensively investigated [13-17] and the lattice Boltzmann method was used to get the numerical solutions for this equations which govern the multi-physical transfort in porous media.

In order to cover the above situations, we employ a spatial time-dependent diffusion coefficient, in other words, our work is aimed at the investigation of solutions for a fractional diffusion equation taking a spatial time-dependence on the diffusion coefficient and an external force (drift) into account. More precisely,we

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focus our attention on the following equation:

$$\frac{\partial^\gamma}{\partial t^\gamma}\rho(x,t) = \int_0^t dt' \frac{\partial}{\partial x}\{D(x,t-t')\frac{\partial^{\mu-1}}{\partial x^{\mu-1}}[\rho(x,t)]^\nu\} - \frac{\partial}{\partial x}\{F(x)\rho(x,t)\}, \tag{1}$$

where $0 < \gamma \leq 1, \mu, \nu \in R$, the diffusion coefficient is given by $D(x,t) = D(t)|x|^{-\theta}$, which is a spatial time-dependent diffusion coefficient,and $F(x) = -\frac{\partial V(x)}{\partial x}$is an external force (drift) associated with the potential $V(x)$. Here, we use the Caputo operator [18] for the fractional derivative with respect to time t, and we work with the positive spatial variable x. Later on, we will extend the results to the entire real x-axis by the use of symmetry (in other words, we are working with $\partial/\partial|x|$ and $\partial^{\mu-1}/\partial|x|^{\mu-1}$). Also, we employ, in general, the initial condition $\rho(x,0) = \tilde{\rho}(x)$ ($\tilde{\rho}(x)$ is a given function), and the boundary condition $\rho(x \to \pm\infty, t) \to 0$. For Eq.(1), one can prove that $\int_{-\infty}^{+\infty} dx\rho(x,t)$ is time independent (hence, if $\rho(x,t)$ is normalized at $t = 0$,it will remain so forever). Indeed, if we write Eq.(1) as $\partial_t^\gamma = -\partial_x J$, and, for simplicity, assume the boundary conditions $J(\pm\infty) = 0$, it can be shown that $\int_{-\infty}^{+\infty} dx\rho(x,t)$ is a constant of motion (see [19] and references therein). Note that when $(\mu,\gamma,\nu) = (2,1,1)$, Eq.(1) recovers the standard Fokker-Planck equation in the presence of a drift taking memory effects into account. The particular case $F(x) = 0$ (no drift) and $D(x,t) = D\delta(t)$ with $(\mu,\gamma) = (2,1)$ has been considered by spohn [20]. The case $D(x,t) = D\delta(t)|x|^{-\theta}$ with $(\mu,\nu) = (2,1)$ and the case $D(x,t) = D(t)$ with $(\mu,\nu) = (2,1)$ have been investigated in [21] and [22], respectively.

Explicit solutions play an important role in analyzing physical situations, since they contain, in principle, precise information about the system. In particular, they can be used as an useful guide to control the accuracy of numerical solutions. For these reasons, we dedicated to this work to investigate the solutions for Eq.(1) in some particular situations. In all the particular cases, Eq.(1) satisfies the initial condition $\rho(x,0) = \tilde{\rho}(x)$ ($\tilde{\rho}(x)$ is a given function), and the boundary condition $\rho(\pm\infty,t) = 0$. The remainder of this paper goes as follow. In Sec.2, we obtain the exact solutions for the special cases. In Sec.3, we present our conclusions.

II. EXACT SOLUTIONS FOR DIFFERENT CASE

In this section, we start our discussion by considering Eq.(1) in the absence of external force with $D(t) = Dt^{\alpha-1}/\Gamma(\alpha)$ ($D(t) = D\delta(t)$),$(\mu,\nu) = (2,1)$ and γ, θ arbitrary. For this case, Eq.(1) reads

$$\frac{\partial^\gamma}{\partial t^\gamma}\rho(x,t) = \int_0^t dt' D(t-t')\frac{\partial}{\partial x}\{|x|^{-\theta}\frac{\partial}{\partial x}\rho(x,t)\}. \tag{2}$$

Here, we use the Caputo operator [18] for the fractional derivative with respect to time t. By employing the Laplace transform in Eq.(2), we obtain

$$\tilde{D}(s)\frac{\partial}{\partial x}\{|x|^{-\theta}\frac{\partial}{\partial x}\tilde{\rho}(x,s)\} - s^\gamma\tilde{\rho}(x,s) = -s^{\gamma-1}\rho(x,0), \tag{3}$$

where $\tilde{\rho}(x,s) = \mathscr{L}\{\rho(x,t)\}$, $\tilde{D}(s) = \mathscr{L}\{D(t)\}$, and $\mathscr{L}\{f(t)\} = \int_0^\infty dte^{-st}f(t)$ denotes the Laplace transform of the function f. This equation can be solved by Green function method [23]. By substituting

$$\tilde{\rho}(x,s) = \int dx'\tilde{\mathcal{G}}(x-x',s)\tilde{\rho}(x'), \tag{4}$$

into Eq.(3) which yields

$$\tilde{D}(s)\frac{\partial}{\partial x}\{|x|^{-\theta}\frac{\partial}{\partial x}\tilde{\mathcal{G}}(x,s)\} - s^{\gamma}\tilde{\mathcal{G}}(x,s) = -s^{\gamma-1}\mathcal{G}(x,0). \tag{5}$$

where $\mathcal{G}(x,t)$ subjects to the initial condition $\mathcal{G}(x,0)=\delta(x)$ and the boundary condition $\mathcal{G}(\pm\infty,t)=0$.

In order to solve Eq.(5), it is convenient to perform the transform [24]

$$y = A(s)x^{v}, \quad \mathcal{G}(x,s) = y^{\delta}Z(y) \tag{6}$$

to translate Eq.(5) into the second-order Bessel equation as

$$y^2\frac{\partial^2 Z}{\partial y^2} + y\frac{\partial Z}{\partial y} - (\lambda^2 + y^2)Z(y) = -\frac{y^{2-\delta}}{s}\delta((\frac{y}{A(s)})^{\frac{1}{v}}) \tag{7}$$

with parameter λ^2 under the following conditions:

$$v = \frac{2+\theta}{2}, A(s) = \frac{1}{v}[\frac{s^{\gamma}}{\tilde{D}(s)}]^{\frac{1}{2}}, \lambda = \frac{1+\theta}{2+\theta}, \delta = \frac{1+\theta}{2+\theta}. \tag{8}$$

Since Eq.(5) should fit the boundary condition $\mathcal{G}(\pm\infty,t)=0$, i.e. $\mathcal{G}(\pm\infty,s)=0$, we get the solution of Eq.(5)

$$\mathcal{G}(x,s) = C(s)y^{\delta}K_{\lambda}(y), \tag{9}$$

where $K_n(x)$ is the modified Bessel function of second kind; and $C(s)$ can be determined by the normalization of $\mathcal{G}(x,t)$, i.e. $\int_0^{\infty} dx\tilde{\mathcal{G}}(x,s) = \frac{1}{2s}$. After some calculations, we obtain

$$\tilde{\mathcal{G}}(x,s) = \frac{2+\theta}{\Gamma(\frac{1}{2+\theta})s}(\frac{1}{2+\theta}(\frac{s^{\gamma}}{\tilde{D}(s)})^{\frac{1}{2}})^{\frac{3+\theta}{2+\theta}}|x|^{\frac{1+\theta}{2}}K_{\frac{1+\theta}{2+\theta}}(\frac{2}{2+\theta}(\frac{s^{\gamma}}{\tilde{D}(s)})^{\frac{1}{2}}|x|^{\frac{2+\theta}{2}}), \tag{10}$$

where, we used the formula

$$\int_0^{\infty} dy \cdot y^{v}K_{\lambda}(ay) = 2^{v-1}a^{-v-1}\Gamma(\frac{1+v+\lambda}{2})\Gamma(\frac{1+v-\lambda}{2}). \tag{11}$$

Case 1. $D(t) = D\delta(t)$, i.e. $\tilde{D}(s) = D$.

Since $K_{\lambda}(x) = \frac{1}{2}H_0^2\ {}_2^0[\frac{x^2}{4}|_{(-\lambda/2,1)(\lambda/2,1)}]$, we can get the Laplace inverse of $\tilde{\mathcal{G}}(x,s)$ by applying the property of the Laplace inverse of Fox function, which yields

$$\mathcal{G}(x,t) = \frac{2+\theta}{2\Gamma(\frac{1}{2+\theta})}(\frac{1}{(2+\theta)^2Dt^{\gamma}})^{\frac{1}{2+\theta}}H_1^2\ {}_2^0[\frac{|x|^{2+\theta}}{(2+\theta)^2Dt^{\gamma}}|^{(1-\frac{\gamma}{2+\theta},\gamma)}_{(0,1),(\frac{1+\theta}{2+\theta},1)}], \tag{12}$$

where $H_p^m\ {}_q^n[x|^{(a_1,A_1),\ldots,(a_p,A_p)}_{(b_1,B_1),\ldots,(b_q,B_q)}]$ is the FOX function [25]. Thus, we can find the solution by substituting Eq.(10) into Eq.(4), which yields

$$\rho(x,t) = \frac{2+\theta}{2\Gamma(\frac{1}{2+\theta})}(\frac{1}{(2+\theta)^2Dt^{\gamma}})^{\frac{1}{2+\theta}}\int_{-\infty}^{+\infty}dx'\tilde{\rho}(x')H_1^2\ {}_2^0[\frac{|x-x'|^{2+\theta}}{(2+\theta)^2Dt^{\gamma}}|^{(1-\frac{\gamma}{2+\theta},\gamma)}_{(0,1),(\frac{1+\theta}{2+\theta},1)}]. \tag{13}$$

In fig.1, we show the behavior of the above equation by considering typical values of γ and θ with $\tilde{\rho}(x)=\delta(x)$. At this point, it is interesting to analyze the asymptotic behavior of Eq.(13). For simplicity, we consider $\tilde{\rho}(x)=\delta(x)$, so $\rho(x,t)=\mathcal{G}(x,t)$; and the asymptotic behavior of $\rho(x,t)$ is

$$\rho(x,t) \sim \frac{2+\theta}{2\Gamma(1/(2+\theta))}(2-\gamma)^{-\frac{1}{2}}\gamma^{\frac{\gamma}{(2+\theta)(2-\gamma)}-\frac{1}{2}}(\frac{1}{(2+\theta)^2Dt^{\gamma}})^{\frac{1}{(2+\theta)(2-\gamma)}}|x|^{\frac{\gamma-1}{2-\gamma}}$$

$$\times exp(-(2-\gamma)\gamma^{\frac{\gamma}{2-\gamma}}(\frac{|x|^{2+\theta}}{(2+\theta)^2Dt^{\gamma}})^{\frac{1}{2-\gamma}}). \tag{14}$$

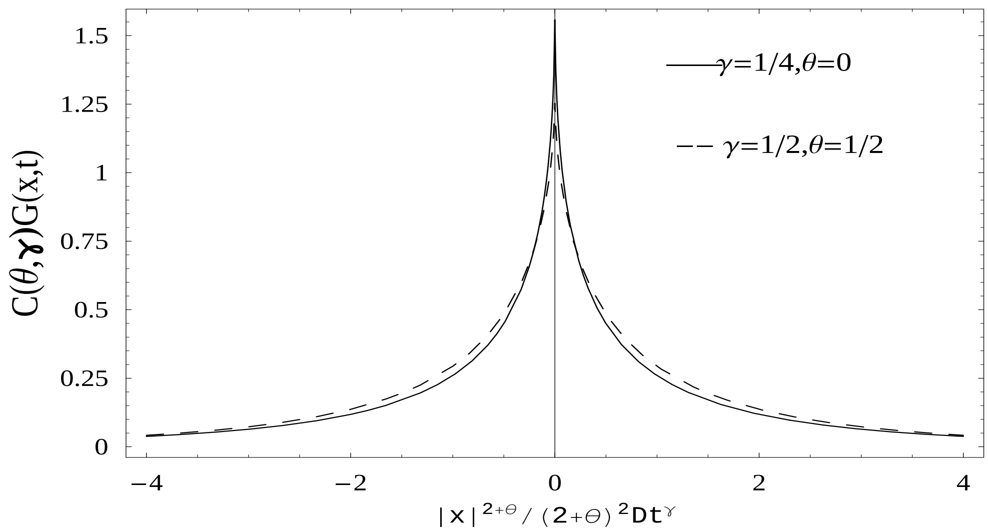

FIG. 1: The behavior of green function $\mathcal{G}(x,t)$ in Eq.(12) is illustrated by considering $C(\theta,\gamma)\mathcal{G}(x,t)$ versus $\frac{|x|^{2+\theta}}{(2+\theta)^2 Dt^{\gamma}}$ for typical values of γ and θ. Here $C(\theta,\gamma)=\frac{2\Gamma(1/(2+\theta))}{2+\theta}((2+\theta)^2 Dt^{\gamma})^{1/(2+\theta)}$.

In this direction, Eq.(14) can be considered as an extension of the asymptotic behavior of homogeneous and isotropic random walk models [26].

Case 2. $D(t)=\frac{Dt^{\alpha-1}}{\Gamma(\alpha)}$, i.e.$\tilde{D}(s)=Ds^{-\alpha}$.

By using the same method as in case 1, we obtain

$$\mathcal{G}(x,t)=\frac{2+\theta}{2\Gamma(\frac{1}{2+\theta})}(\frac{1}{(2+\theta)^2 Dt^{\gamma+\alpha}})^{\frac{1}{2+\theta}} H_1^2 \;\; {}_2^0[\frac{|x|^{2+\theta}}{(2+\theta)^2 Dt^{\gamma+\alpha}}|^{(1-\frac{\gamma+\alpha}{2+\theta},\gamma+\alpha)}_{(0,1),(\frac{1+\theta}{2+\theta},1)}], \tag{15}$$

and

$$\rho(x,t)=\frac{2+\theta}{2\Gamma(\frac{1}{2+\theta})}(\frac{1}{(2+\theta)^2 Dt^{\gamma+\alpha}})^{\frac{1}{2+\theta}} \int_{-\infty}^{+\infty} dx'\tilde{\rho}(x') H_1^2 \;\; {}_2^0[\frac{|x-x'|^{2+\theta}}{(2+\theta)^2 Dt^{\gamma+\alpha}}|^{(1-\frac{\gamma+\alpha}{2+\theta},\gamma+\alpha)}_{(0,1),(\frac{1+\theta}{2+\theta},1)}]. \tag{16}$$

Let us go back to Eq.(1), and consider the external force $F(x)\propto x|x|^{\alpha-1}$, $D(x,t)=D\delta(t)|x|^{-\theta}$ and $\mu=2$, $\nu=1$. In this case ,analytical solution can not easily be obtained for a generic α, θ. However, for $\theta\neq 0$, and $\alpha+\theta+1=0$. By following the same procedure as in the above case, an exact solution can be obtained and it is given by

$$\rho(x,t)=\frac{2+\theta}{2\Gamma(\frac{1}{2+\theta}+\frac{\mathcal{K}}{(2+\theta)D})}(\frac{1}{(2+\theta)^2 Dt^{\gamma}})^{\frac{1}{2+\theta}} H_1^2 \;\; {}_2^0[\frac{|x|^{2+\theta}}{(2+\theta)^2 Dt^{\gamma}}|^{(1-\frac{\gamma}{2+\theta},\gamma)}_{(\frac{\mathcal{K}}{(2+\theta)D},1),(\frac{1+\theta}{2+\theta},1)}], \tag{17}$$

where, for simplicity, we are considering the initial condition $\rho(x,0)=\delta(x)$, and the external force (drift) $F(x)=\mathcal{K}x^{\alpha}$. The second moment is given by $<x^2>\propto t^{\frac{2\gamma}{2+\theta}}$, which scales with the exponent $\frac{2\gamma}{2+\theta}$ and clearly depends only on γ and θ. So, when $\frac{2\gamma}{2+\theta}<1$, $=1$ and >1, the system is sub-diffusion, normal diffusion and supper-diffusion respectively.

The presence of the external force in Eq.(1) is now changed into $F(x)=-k_1x+k_2x^{-1-\theta}$. In order to obtain the solution of Eq.(1), we expand $\rho(x,t)$ in terms of the eigenfunctions, i.e. we employ $\rho(x,t)=$

$\sum_{n=0}^{\infty}\phi_n(t)\psi_n(x)$ with $\psi_n(x)$ determined by the spatial equation

$$-\lambda_n\psi_n(x) = D\frac{\partial}{\partial x}\{x^{-\theta}\frac{\partial}{\partial x}\psi_n(x)\} + \frac{\partial}{\partial x}\{(k_1x + k_2x^{-1-\theta})\psi_n(x)\} \tag{18}$$

and $\phi_n(t)$ determined by the time equation

$$\frac{\partial^\gamma}{\partial t^\gamma}\phi_n(t) = -\lambda_n\phi_n(t). \tag{19}$$

The solution for the time equation is given by in terms of the Mittag-Leffter function

$$\phi(t) \propto E_\gamma(-\lambda_n t^\gamma). \tag{20}$$

In order to get the solution for Eq.(18), we perform the transform

$$\psi_n(x) = e^{-y}y^\delta Z(y), \quad y = Ax^v \tag{21}$$

to translate Eq.(18) into the associated Laguerre equation

$$yZ''(y) + (\alpha + 1 - y)Z'(y) + nZ(y) = 0 \tag{22}$$

with parameter α under the following condition:

$$\alpha = \frac{\frac{k_2}{D} - 1 - \theta}{2+\theta}, v = 2+\theta, A = \frac{k_2}{(2+\theta)D}, \delta = \frac{k_2}{(2+\theta)D}, \lambda_n = (2+\theta)nk_1. \tag{23}$$

Then, using the Green function methods and after some calculations, it is possible to show that

$$\rho(x,t) = \int_{-\infty}^{\infty} dx_0\tilde{\rho}(x_0)\mathcal{G}(x,x_0,t), \tag{24}$$

$$\mathcal{G}(x,x_0,t) = (\frac{k_1}{(2+\theta)D})^{\frac{k_2+D}{(2+\theta)D}}|x|^{\frac{k_2}{D}}e^{-\frac{k_1}{(2+\theta)D}|x|^{2+\theta}}\sum_{n=0}^{\infty}\frac{(2+\theta)\Gamma(n+1)}{\Gamma(n+\frac{k_2+D}{(2+\theta)D})}E_\gamma(-\lambda_n t^\gamma)$$

$$\times L_n^{(\alpha)}(\frac{k_1}{(2+\theta)D}|x|^{2+\theta})L_n^{(\alpha)}(\frac{k_1}{(2+\theta)D}|x_0|^{2+\theta}), \tag{25}$$

where $L_n^{(\alpha)}(x)$ is the associated Laguerre polynomial and it is the solution for Eq.(22). Here, we used the formula

$$\int_0^\infty dyy^\alpha e^{-y}L_n^{(\alpha)}(y)L_n^{(\alpha)}(y) = \frac{\Gamma(n+\alpha+1)}{\Gamma+1}. \tag{26}$$

Notice that Eq.(25) contains the usual Ornstein-Uhlenbeck process [27] and the usual Rayleigh process [28] as particular cases and it extends the results obtained in [21]. In this context, in the presence of an constant absorbent force, i.e. $\tilde{\alpha}\rho(x,t)$, we can obtain the solution for Eq.(1) only need to change the argument of the Mittag-Leffler function present in Eq.(20) to $\lambda_n + \tilde{\alpha}$.

Let us now discuss Eq.(1) by considering a mixing between the spatial and time fractional derivatives. For simplicity, we consider Eq.(1) in the absence of the external force with $(\nu,\theta) = (1,0)$ and γ, μ arbitraries.

Applying the Fourier and Laplace transform to Eq.(1) and employing the Riez representation for the spatial fractional derivatives, we have

$$s^{\gamma}\hat{\tilde{\rho}}(k,s)-s^{r-1}\hat{\rho}(k,0)=-\tilde{D}(s)|k|^{\mu}\hat{\tilde{\rho}}(k,s), \tag{27}$$

where $\hat{\rho}(k,t)=\mathcal{F}\{\rho(x,t)\}=\int_{-\infty}^{+\infty}\rho(x,t)e^{-ikx}dx$, so $\rho(k,0)=1$. Here, we consider the diffusion coefficient given by $D(t)=\frac{Dt^{\alpha-1}}{\Gamma(\alpha)}$, i.e. $\tilde{D}(s)=Ds^{-\alpha}$ and $\rho(x,0)=\delta(x)$. By using the inverse of Laplace transform, we obtain

$$\rho(k,t)=E_{\gamma+\alpha,1}(-D|k|^{\mu}t^{\gamma+\alpha})$$

$$=H_{1\quad 2}^{1\quad 1}[D|k|^{\mu}t^{\gamma+\alpha}|_{(0,1),(0,\gamma+\alpha)}^{(0,1)}]. \tag{28}$$

This solution recovers the usual one for $(\mu,\gamma)=(2,1)$ and for $\mu\neq 2$ it extends the results found in [29]. In order to perform the inverse of Fourier transform, we employ the procedure presented in [30]. Then we can obtain

$$\rho(x,t)=\frac{1}{2\mu\sqrt{\pi}(Dt^{\gamma+\alpha})^{\frac{1}{\mu}}}H_{2\quad 3}^{2\quad 1}[\frac{|x|}{2(Dt^{\gamma+\alpha})^{\frac{1}{\mu}}}|_{(0,1/2),(1-1/\mu,1/\mu),(1/2,1/2)}^{(1-1/\mu,1/\mu),(1-(\gamma+\alpha)/\mu,(\gamma+\alpha)/\mu)}]. \tag{29}$$

The stationary solution that emerges from this process is a Levy distribution.

Now, we consider a particular case of Eq.(1) for $\gamma=1$ and nonzero values of μ and θ, and consider a linear drift,i.e. $F(x)=-\mathcal{K}x$. For simplicity, we employ $D(t)=D\delta(t)$ and the initial condition $\rho(x,0)=\delta(x)$, then Eq.(1) yields to

$$\frac{\partial}{\partial t}\rho(x,t)=D\frac{\partial}{\partial x}\{|x|^{-\theta}\frac{\partial^{\mu-1}}{\partial x^{\mu-1}}[\rho(x,t)]^{\nu}\}+\frac{\partial}{\partial x}\{\mathcal{K}x\rho(x,t)\}. \tag{30}$$

Let us investigated time dependent solutions for Eq.(30). We use similarity methods to reduce Eq.(30) to ordinary differential equations. The explicit form for these ordinary differential equations depends on the boundary conditions or restrictions in the form of conservation laws. In this direction, we restrict our analysis to find solution that can be expressed as a scaled function of the type [31]

$$\rho(x,t)=\frac{1}{\phi(t)}\tilde{\rho}(z),\quad z=\frac{x}{\phi(t)}. \tag{31}$$

Inserting Eq.(31) into Eq.(30), we obtain

$$-(\frac{\dot{\phi(t)}}{\phi(t)^{2}}+\frac{\mathcal{K}}{\phi(t)})\frac{\partial}{\partial z}[z\tilde{\rho}(z)]=\frac{D}{\phi(t)^{\theta+\mu+\nu}}\frac{\partial}{\partial z}[z^{-\theta}\frac{\partial^{\mu-1}}{\partial z^{\mu-1}}\tilde{\rho}(z)^{\nu}]. \tag{32}$$

By choosing the ansatz

$$\frac{\dot{\phi(t)}}{\phi(t)^{2}}+\frac{\mathcal{K}}{\phi(t)}=\frac{kD}{\phi(t)^{\theta+\mu+\nu}}, \tag{33}$$

where k is an arbitrary constant which can be determined by the normalization condition. By solving Eq.(33), we have that

$$\phi(t)=[(\phi(0))^{\theta+\mu+\nu-1}e^{-(\theta+\mu+\nu-1)\mathcal{K}t}+\frac{Dk}{\mathcal{K}}(1-e^{-(\theta+\mu+\nu-1)\mathcal{K}t})]^{\frac{1}{\theta+\mu+\nu-1}}. \tag{34}$$

By substituting Eq.(33) into Eq.(32), we obtain

$$\frac{\partial}{\partial z}[z^{-\theta}\frac{\partial^{\mu-1}}{\partial z^{\mu-1}}\tilde{\rho}(z)^{\nu}] = -k\frac{\partial}{\partial z}[z\tilde{\rho}(z)]. \tag{35}$$

Then, we perform an integration and the result is

$$z^{-\theta}\frac{\partial^{\mu-1}}{\partial z^{\mu-1}}\tilde{\rho}(z)^{\nu} = -kz\tilde{\rho}(z) + \mathcal{C}, \tag{36}$$

where $\mathcal{C}$ is another arbitrary constant. Also, we use the following generic result [32]:

$$D_x^{\delta}[x^{\alpha}(a+bx)^{\beta}] = a^{\delta}\frac{\Gamma[\alpha+1]}{\Gamma[\alpha+1-\delta]}x^{\alpha-\delta}(a+bx)^{\beta-\delta} \tag{37}$$

with $D_x^{\delta} \equiv d^{\delta}/dx^{\delta}$ and $\delta = \alpha+\beta+1$. By defining $g(x) \equiv x^{\frac{\alpha}{\nu}}(a+bx)^{\frac{\beta}{\nu}}$ and $\lambda \equiv \alpha(1-\frac{1}{\nu})-\delta$, and rearranging the indices, Eq.(37) can be rewritten as follows:

$$D_x^{\delta}[g(x)]^{\nu} = a^{\delta}\frac{\Gamma[\alpha+1]}{\Gamma[\alpha+1-\delta]}x^{\lambda}g(x). \tag{38}$$

For this case, we consider the ansatz $\tilde{\rho}(z) = \mathcal{N}z^{\frac{\alpha}{\nu}}(1+bz)^{\frac{\beta}{\nu}}$. By using the property of Eq.(38) in Eq.(36) and ,for simplicity, choosing $\mathcal{C} = 0$, we find

$$\alpha = \frac{(2-\mu)(\mu+\theta)}{1-2\mu-\theta},$$

$$\beta = -\frac{(\mu-1)(\mu-2)}{1-2\mu-\theta}, \tag{39}$$

$$\nu = \frac{2-\mu}{1+\mu+\theta}.$$

In this case, we have

$$\rho(x,t) = \frac{\mathcal{N}}{\phi(t)}[\frac{z^{(\mu+\theta)(1+\mu+\theta)}}{(1+bz)^{(1-\mu)(1+\mu+\theta)}}]^{\frac{1}{1-2\mu-\theta}}, \tag{40}$$

where $\phi(t)$ is given above, $\mathcal{N} = [-k\frac{\Gamma(-\beta)}{\Gamma(\alpha+1)}]^{\frac{\mu+\theta+1}{1-2\mu-\theta}}$ and b is an arbitrary constant (to be taken, later on, as ± 1 according to the specific solutions that are studied). Several regions can be analyzed. For simplicity, we illustrate two of them: $-\infty < \mu < -1-\theta$ with $\theta \geq 0$, and $0 < \mu < 1/2$ with $0 \leq \theta < 1/2-\mu$. Let us start by considering the region $-\infty < \mu < -1-\theta$. Without loss of generality, we choose $b = -1$. The normalization condition implies(see Fig. 2)

$$\mathcal{N}\int_{-1}^{1}[\frac{z^{(\mu+\theta)(1+\mu+\theta)}}{(1+bz)^{(1-\mu)(1+\mu+\theta)}}]^{\frac{1}{1-2\mu-\theta}}dz = 1. \tag{41}$$

So

$$\mathcal{N} = \frac{\Gamma[1-\mu-\theta]}{2\Gamma[\frac{\mu^2+\mu\theta-2\theta-2\mu}{1-2\mu-\theta}]\Gamma[\frac{1-\mu+\mu^2+\theta^2+2\mu\theta}{1-2\mu-\theta}]}. \tag{42}$$

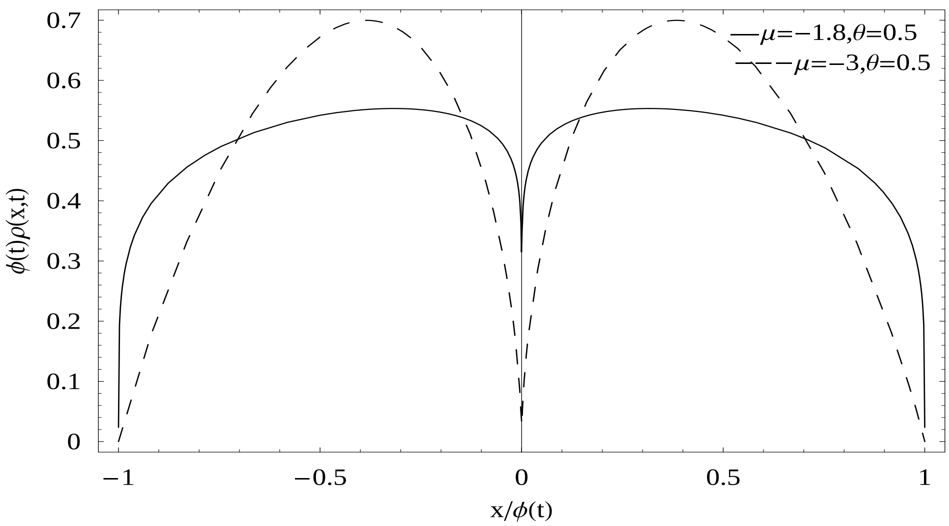

FIG. 2: Behavior of $\phi(t)\rho(x,t)$ versus $x/\phi(t)$, which illustrates Eq.(40) with typical values for μ and θ satisfying $0 < \mu < -1-\theta$ and $\theta \geq 0$. We notice that the distribution vanishes at the abcissa equal ± 1, and remains zero outside of this interval.

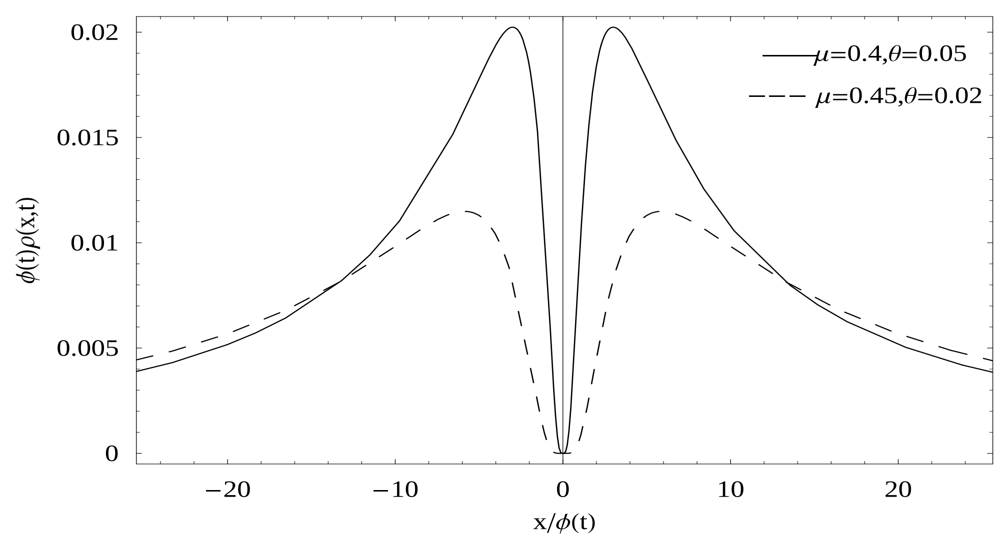

FIG. 3: Behavior of $\phi(t)\rho(x,t)$ versus $x/\phi(t)$, which illustrates Eq.(40) with typical values for μ and θ satisfying $0 < \mu < 1/2$ and $0 \leq \theta < 1/2 - \mu$. We notice that the distribution vanishes at the infinity.

Let us now analyze the region $0 < \mu < 1/2$ with $0 \leq \theta < 1/2 - \mu$. Again without the loss of generality, we choose $b = 1$. The normalization condition implies(see Fig. 3)

$$\mathcal{N} = \frac{\Gamma[\frac{1+\theta-\mu^2-\mu\theta}{1-2\mu-\theta}]}{2\Gamma[\frac{1-\mu+\mu^2+\theta^2+2\mu\theta}{1-2\mu-\theta}]\Gamma[\mu+\theta]}. \tag{43}$$

Let us finally mention a connection between the results obtained here and the solutions that arise from the optimization of the nonextensive entropy [33]. These distributions do not coincide for arbitrary value of x. However, the comparison of the $|x| \to \infty$ asymptotic behaviors enables us to identify the type of Tails.

By identifying the behavior exhibited in Eq.(40) with the asymptotic behaviors $1/|x|^{2/(q-1)}$ that appears in [33] for the entropic problem, we obtain

$$q = \frac{3+\mu+\theta}{1+\mu+\theta}. \tag{44}$$

This relation recovers the situation for $\theta = 0$.

III. SUMMARY AND CONCLUSIONS

We have analyzed the generalized fractional diffusion equations by considering an external force $F(x) \propto x|x|^{\alpha-1}$ and a spatial time-dependent diffusion coefficient $D(x,t) = D(t)|x|^{-\theta}$. By using Laplace transform, Fourier transform, the Green function method and normalized scaled function we can find the explicit solutions $\rho(x,t)$ which subjects to the natural boundary condition $\rho(\pm\infty,t) = 0$ and the initial condition $\rho(x,t) = \tilde{\rho}(x)$. In a word, we have extended the results previously obtained by the other authors by taking an external force and a spatial time-dependent diffusion coefficient into account. We have also discussed the connection with nonextensive statistics, providing the relation between our solutions and those obtained within the maximum entropy principle by using the Tsallis entropy. Finally, we expect that the results obtained here may be useful to the discission of the anomalous diffusion systems where fractional diffusion equations play an important role.

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